and $\mathbb{A}$ is the value of an annuity of £1 payable yearly, calculated at the true annual rate of interest $\left(1+\frac{i}{m}\right)^m-1=j$;

$$\text{or } A^{(m)}=\frac{1}{i^2}\left(\frac{m+i}{m}\right)^{m+1}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^2\mathbb{A}+\frac{1}{i}-\frac{m+i}{mi^2}\left\{1-\left(\frac{m}{m+i}\right)^m\right\}^*$$

$$\text{or } a_k^{(m)}=\left\{1+\left(1-\frac{1}{m^2}\right)\frac{j^2}{12}+\ldots\right\}a_k+\frac{m-1}{2m}-\frac{m^2-1}{6m^2}j+\ldots^*$$

In the last three formulæ, but in none of the others here given, i is the nominal rate of interest convertible m times a year, and j is the true annual interest.

$$\text{Woolhouse, } a_k^{(m)}=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}(\mu+\delta)$$

$$=a_k+\frac{m-1}{2m}-\frac{m^2-1}{12m^2}\left(\frac{l_{k-1}-l_{k+1}}{2l_k}+i\right)^*$$

Errata in Part I.

Page 190, line 3 from bottom, for $\frac{3}{32}i^3$ read $\frac{3}{32}i^2$.

„ 192, „ 19, for $(m+1)^x$ read $(m+i)^x$.

„ 193, „ 16, for Σ^t read Σ_t.

„ „ throughout, for p_{k+t} read $p_{k,t}$.

„ „ „ for p_{k+t-1} read $p_{k,t-1}$.

„ „ line 20, for "value" read "amount".

„ 195, „ 6 from bottom, for $\frac{640+160i+20i^2+i^3}{1024}$ read $\frac{96+32i+3i^2}{4(4+i)^3}$.

On Annuities and Assurances on Successive Lives. By Thomas Weddle, F.R.A.S.*

MILNE, in his *Treatise on Annuities*, chapter vii., has discussed this subject with much clearness and elegance; yet his method of investigation does not seem to be the best possible, nor do I think that his results are given in forms well adapted to numerical applications. It may therefore be not altogether useless to consider the subject in a somewhat different manner, and so to exhibit the formulas as best to meet the wants of the computer.

PROBLEM I.—To determine the present value (a_n) of an assurance of £1 payable on the failure of the last of n successive lives $A', A'' \ldots . A^{(n)}$.

* This paper appeared in the *Philosophical Magazine* for January, 1850, and is the one referred to by Mr. Peter Gray, at page 1, vol. ii., of this *Journal*.—Ed. *J. I. A.*

Let $a^{(p)}$ denote the value at the time of nomination of an assurance of £1 payable on the death of $A^{(p)}$; $A^{(p)}$ itself denoting the value of an annuity of £1 during the life of $A^{(p)}$, so that we have (r being the interest of £1 for a year)

$$a^{(p)}=\frac{1-rA^{(p)}}{1+r}; \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

$$\therefore\ 1-a^{(p)}=\frac{r}{1+r}(1+A^{(p)}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and

$$1+A^{(p)}=\left(1+\frac{1}{r}\right)(1-a^{(p)}). \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We have evidently $a_{,}=a'$; also when A'' shall be nominated the value of £1 payable at his death will be a''; but the present value of £1 payable when A'' shall be nominated, that is, at the end of the year in which A' shall die, is $a_{,}$; hence the present value of £1 payable at the death of A'' is

$$a\times a''=a'a'';$$
$$\therefore\ a_2=a'a''.$$

And generally when the life $A^{(p)}$ shall be nominated, the then value of £1 payable at his death will be $a^{(p)}$; but the present value of £1 payable at the death of $A^{(p-1)}$, that is, at the nomination of $A_{(p)}$, is a_{p-1},

$$\therefore\ a_p=a_{p-1}\,.\,a^{(p)}.$$

Hence taking $p=1, 2, 3 \ldots . n$ in succession, we have

$$\begin{aligned} a_1&=a' \\ a_2&=a_1a'' \\ a_3&=a_2a''' \\ &\vdots \\ a_n&=a_{n-1}\,.\,a^{(n)}. \end{aligned}$$

Multiply these equations together and cancel the common factors

$$a_n=a'a''a''' \ldots . a^{(n)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence the present value of an assurance payable on the failure of the last of any number of successive lives is very readily computed, provided we have a table of the present values of assurances on single lives; and such tables are given in David Jones's work on *Annuities and Reversionary Payments* (Tables IX. and XXII.), according to both the Northampton and Carlisle tables of mortality. If however only a table of annuities be at hand, it will be better to modify (4) as follows:—

By (1) we have $a^{(p)}=v(1-rA^{(p)})$, where $v=(1+r)^{-1}$, the present value of £1, due in a year, hence (4) becomes

$$a_n=v^n(1-rA').(1-rA'')\ldots(1-rA^{(n)}). \quad . \quad . \quad . \quad . \quad (5)$$

Problem II.—To determine (a_p) the present value of an annuity of £1 on the pth life mentioned in Problem I.

At the nomination of $A^{(p)}$ £1 will be due, and the annuity on his life will be worth $A^{(p)}$; hence the life will then be worth $1+A^{(p)}$; but £1 due at the nomination of $A^{(p)}$ is now worth a_{p-1}; hence the present value of the annuity on the pth life is

$$a_{p-1}.(1+A^{(p)}),$$

$$\therefore, (4), \; a_p=a'a''\ldots a^{(p-1)}.(1+A^{(p)}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

and this is the present value of £1 per annum on the pth life.

The preceding (6) is the form that will generally be found best adapted to computation; but a_p may also be expressed in terms of a only, or of A only, as follows (see (1) and (3)):—

$$a_p=\left(1+\frac{1}{r}\right)a'a''\ldots a^{(p-1)}(1-a^{(p)}), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

and

$$a_p=v^{p-1}(1-rA')(1-rA'')\ldots(1-rA^{(p-1)})(1+A^{(p)}) \quad . \quad (8)$$

It may be observed too that (7) is little, if any, inferior to (6) in point of easy application, as it requires a reference to one table only, while (6) requires a reference to two.

Note.—Unless the annuity be *due*—that is, unless a payment is to be made immediately—none of the preceding formulas will give a_1, the value of the annuity on the first life, correctly. If we suppose, as usual, that the first payment will be made in a year, the true value of a_1 will be

$$a_1=A'=\left(1+\frac{1}{r}\right)(1-a')-1, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

and not

$$a_1=1+A'=\left(1+\frac{1}{r}\right)(1-a').$$

Problem III.—To determine (A_n), the present value of an annuity of £1 on the n successive lives $A', A'' \ldots A^{(n)}$.

We evidently have

$$A_n=a_1+a_2+a_3\ldots+a_n.$$

But (7) and (9)—

$$a_1=\left(1+\frac{1}{r}\right)(1-a')-1$$

$$a_2=\left(1+\frac{1}{r}\right)(a'-a'a'')$$

$$a_3 = \left(1 + \frac{1}{r}\right)(a'a'' - a'a''a''')$$

$$\vdots$$

$$a_n = \left(1 + \frac{1}{r}\right)(a'a'' \ldots . a^{(n-1)} - a'a'' \ldots . a^{(n)}).$$

Hence by addition we have

$$A_n = \left(1 + \frac{1}{r}\right)(1 - a'a'' \ldots . a^{(n)}) - 1 \quad . \quad . \quad . \quad . \quad (10)$$

It appears from (4) that (10) is equivalent to

$$A_n = \left(1 + \frac{1}{r}\right)(1 - a_n) - 1;$$

and, in fact, this might have been deduced immediately from (3), for the formula (1) which expresses the relation between an annuity and the corresponding assurance, is true of any status*, providing the status be such that the assurance on it *must* be paid some time or other.

PROBLEM IV.—A copyhold estate is held on a certain number of lives A, B, C . . . . and each life is renewable at the end of the year in which it may fail by paying a fine of f pounds. Required the present value of all the fines.

Let A′, A″ . . . . be the lives that succeed A; B′, B″ . . . . those that succeed B, &c.; also let a, a′ a″ . . . . be the values of assurances on the lives A, A′, A″ . . . .; b, b′, b″ . . . . those on B, B′, B″ . . . ., &c.

By (4) it appears that the fines (each equal to f) payable at the deaths of A, A′, A″ . . . . are now worth fa, $f$aa′, $f$aa′a″ . . . . respectively; and similar expressions being true of the other successive lives, the present value of all the fines is

$$f \times \left\{ \begin{array}{l} \ \ a + aa' + aa'a'' + \ldots\ldots \\ + b + bb' + bb'b'' + \ldots\ldots \\ + c + cc' + cc'c'' + \ldots\ldots \\ + \ \ldots\ldots\ldots\ldots\ldots\ldots \end{array} \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

If the lives, at the time of the nomination, be all of the same age (which must be assumed in practice), then, P denoting the value at the time of nomination of an annuity of £1 on each renewal life, and p the assurance on the same, we shall have

$$a' = a'' = \ldots . = b' = b'' = \ldots\ldots = c' = c'' = \ldots\ldots.. = p,$$

* "By the *status* of an annuity, I mean the state or condition of things during the continuance of which the annuity is to be paid."—De Morgan's *Essay on Probabilities*, p. 190.

and each horizontal row in (11) now constitutes a geometrical progression of which the common ratio is p; and the sum of the whole is evidently

$$f.\left\{\frac{\mathrm{a}}{1-\mathrm{p}}+\frac{\mathrm{b}}{1-\mathrm{p}}+\frac{\mathrm{c}}{1-\mathrm{p}}+\ldots.\right\}$$

Hence

$$f.\frac{\mathrm{a}+\mathrm{b}+\mathrm{c}+\ldots\ldots\ldots}{1-\mathrm{p}}, \quad . \ . \ . \ . \ . \ . \ . \quad (12)$$

is the present value of all the fines.

The preceding expression (12) may also be exhibited in terms of the annuities on the lives instead of the assurances on the same. For

(1), $\mathrm{a}=\dfrac{1-r\mathrm{A}}{1+r}$, $\mathrm{b}=\dfrac{1-r\mathrm{B}}{1+r}$, ..., and, (2), $1-\mathrm{p}=\dfrac{r}{1+r}(1+\mathrm{P})$,

hence (12) becomes

$$f.\frac{n\dfrac{1}{r}-(\mathrm{A}+\mathrm{B}+\mathrm{C}+\ldots.)}{1+\mathrm{P}}, \quad . \ . \ . \ . \ . \ . \quad (13)$$

where n denotes the number of lives A, B, C ... on which the estate is held, and it may be observed that $\frac{1}{r}$ is a perpetuity of £1.

The formulas (6), (10), (11) and (12), agree with those given by Mr. Milne, but they are here expressed in a way better adapted to computation. That these coincide with Mr. Milne's expressions, will readily appear, if, instead of A′ ..., we introduce annuities certain of equal values; thus let $\mathrm{A}^{(p)}$ be equivalent to an annuity certain for t_p years, so that

$$\mathrm{A}^{(p)}=\frac{1-v^{t_p}}{r},$$

$$\therefore\ 1+\mathrm{A}^{(p)}=\left(1+\frac{1}{r}\right)(1-v^{t_p+1}),\ \text{and, (2), }\ \mathrm{a}^{(p)}=v^{t_p+1},$$

hence (10) becomes

$$\mathrm{A}_n=\left(1+\frac{1}{r}\right)(1-v^{\sigma+1})-1,\ (\sigma=n-1+t_1+t_2\ldots.+t_n),$$

or

$$\mathrm{A}_n=\frac{1-v^{\sigma}}{r},$$

which is Mr. Milne's formula; and similarly, it may be shown that (6) and (11) coincide with the expressions given in Milne.

Also if P be equivalent to an annuity certain of t years, so that $P=\frac{1-v^t}{r}$, and therefore $p=v^{t+1}$, then (12) becomes

$$f\frac{a+b+c+\ \ldots}{1-v^{t+1}}.$$

which is Mr. Milne's expression.

The formula (13) is not in Milne, but is given by Professor De Morgan in his *Essay on Probabilities* (*Cab. Cyclop.*), Appendix the Second. It is very well adapted to computation, though not I think quite so well as (12), if Jones's tables previously alluded to be used. The difference between the two formulas in point of practical application is, however, very trifling.

Eighth Census of the United States, in 1860. *By* SAMUEL BROWN, *Esq.*, V.P.S.S.

IN the course of last year a very important document was published by the American Government, under the direction of the Secretary of the Interior, entitled "Statistics of the United States (including Mortality, Property, &c.) in 1860; compiled from the original Returns, and being the final Exhibit of the Eighth Census." It contains an introduction by J. M. Edmunds, Commissioner of General Land Office, in charge of census, and a still fuller introduction to the mortality statistics for the year ending June 1st, 1860, by Dr. Edward Jarvis, a very able writer, who was a delegate to the Statistical Congress, when it was held in London, and is a Corresponding Member of the Statistical Society of London.

The former introduction gives a brief notice of the census as taken in different countries (commencing with the enumeration of the people under the Mosaic Dispensation) to the first census of the United States, in 1790. Since that time there has been one every 10 years, and a short sketch is given of the legal enactments by which they were carried out. The statute of 23rd May, 1850, by which the seventh census was ordered, also made provision for the eighth and any subsequent census, which was supplemented in 1860 and 1862 by some Acts which required, amongst other things, the Secretary of War to be furnished with such war statistics as might be needed, and gave a credit of two millions of dollars =£400,000 for the necessary expenses, including the costs of printing and binding.